＜研究ノート＞

# Glauberman correspondence as a Brauer construction

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**Abstract**

We introduce a particular case of Dade's interpretation of the Glauberman correspondence in terms of the modular representation theory of finite groups, and remark that the Glauberman correspondence can be viwed as a module correspondence given by the Brauer construction of the module.

*Keywords*: finite group; Glauberman correspondence; Brauer construction; Brauer correspondence; Green correspondence

## 1

For a prime $p$, let $(\mathcal{K}, \mathcal{O}, k)$ be a $p$-modular system where $\mathcal{O}$ is a complete discrete valuation ring having an algebraically closed residue field $k$ of characteristic $p$ and having a quotient field $\mathcal{K}$ of characteristic zero which will be assumed to be large enough for any of finite groups we consider in this article. Let $G$ be a finite group.

For the standard facts on the modular representation theory of finite groups, see [1], [4] and [5].

## 2

In this section, we give a result on the Brauer construction of the module.

**Proposition 2.1.** *Let $P$ be a Sylow $p$-subgroup of $G$, and assume that $P$ has an order $p$. Let $V$ be a simple $kG$-module with a vertex $P$. Then $V(P)$, the Brauer construction of $V$ viewed as a $kN_G(P)$-module, is simple and is isomorphic to the socle of the Green correspondence of $V$.*

**Proof**

By the Green coreespondence in a T.I. situation, we have $V{\downarrow}^G_{N_G(P)} \simeq W \oplus Y$ where $W$ is an indecomposable module with a vertex $P$ and $Y$ is a projective module, see [1, Theorem 1 of Chapter10].

Note that $Y(P) = 0$, see [5, Proposition 27.9].

We have $\mathrm{Tr}_1^P(W) = 0$. In fact, since the indecomposable projective $kN_G(P)$-modules are uniserial, see [1, the last part of Chapter 5, or Theorem 1 of Chapter 19], we have $W \subseteq \mathrm{rad}R$ for an indecomposable projective $kN_G(P)$-module $R$. Since $\mathrm{rad}R = (1-u)R$ for a generator $u$ of $P$, see [1, Lemma 8 of Chapter 5], we have $(1+u+\cdots+u^{p-1})\,\mathrm{rad}R = 0$. Hence we have $(1+u+\cdots+u^{p-1})W = 0$ and the assertion follows.

Hence we have

$$\begin{aligned} V(P) &\simeq \left(V{\downarrow}^G_{N_G(P)}\right)(P) \\ &\simeq (W \oplus Y)(P) \\ &\simeq W(P) \oplus Y(P) \\ &\simeq W(P) \\ &\simeq W^P/\mathrm{Tr}_1^P(W) \\ &\simeq W^P. \end{aligned}$$

We have

$$W^P = \mathrm{soc}(W{\downarrow}_P^{N_G(P)}),$$

since for an indecomposable $kP$-module X we have $X^P = \mathrm{soc}\, X$. In fact, we have $\mathrm{soc}\, X \subseteq X^P$ and $1 = \dim_k \mathrm{soc}\, X \leq \dim_k X^P \leq \dim_k (kP)^P = 1$. Note that if $W{\downarrow}_P = \oplus_i X_i$, then $W^P = (\oplus_i X_i)^P = \oplus_i X_i^P$ and $\mathrm{soc}(W{\downarrow}_P) = \mathrm{soc}(\oplus_i X_i) = \oplus_i \mathrm{soc}(X_i)$.

Moreover, using [1, Lemma 8 of Chapter 5] recursively, we have

$$\mathrm{soc}(W{\downarrow}_P^{N_G(P)}) = \mathrm{soc}(W).$$

Hence, we have the proposition. □

# 3

In this section, we introduce a particular case of Dade's interpretation of the Glauberman correspondence (see [3]) in terms of the modular representation theory of finite groups (see Section 13 of [2] for a more general statement). Then we remark that the Glauberman correspondence can be viwed as a module correspondence given by the Brauer construction of the module.

We cite a very particular case of Dade's Theorem on the endo-permutation modules, which is essential in the proof of Proposition 3.3:

**Proposition 3.1.** *(Dade [2] or see [5, Theorem 30.5, Proposition 28.2, Corollary 28.11])*

*(i) The sources of simple modules of $p$-nilpotent groups are endo-permutation modules.*

*(ii) Endo-permutation $kP$-modules for a group $P$ of order $p$ are $k$ and $\Omega_k(k)$.*

Below, we assume that $G$ has an order not divisible by $p$. Let $P$ be a group of order $p$ acting on $G$, and $E$ be a semidirect product of $G$ and $P$ with this action. Let $C = C_G(P)$. We have $N_E(P) = C_E(P) = CP$.

The following is standard, and below we consider the correspondence suggested in the proof of Lemma 3.2:

**Lemma 3.2.** *There is a one-to-one correspondence between the following sets:*

*(i)* $\mathrm{Irr}(G)^P$*: the set of $P$-invariant irreducible characters of $G$*

*(ii)* $\mathrm{Simp}(G)^P$*: the set of $P$-invariant simple $kG$-modules*

*(iii)* $\mathrm{Bl}(G)^P$*: the set of $P$-invariant $p$-blocks of $G$*

*(iv)* $\mathrm{Bl}(E|P)$*: the set of $p$-blocks of $E$ with a defect group $P$*

*(v)* $\mathrm{Simp}(E|P)$*: the set of simple $kE$-modules with a vertex $P$*

*(vi)* $\mathrm{Bl}(CP|P)$*: the set of $p$-blocks of $CP$ with a defect group $P$*

*(vii)* $\mathrm{Simp}(CP|P)$*: the set of simple $kCP$-modules with a vertex $P$*

*(viii)* $\mathrm{Bl}(C)^P$*: the set of $P$-invariant $p$-blocks of $C$*

*(ix)* $\mathrm{Bl}(C)$*: the set of $p$-blocks of $C$*

*(x)* $\mathrm{Simp}(C)$*: the set of simple $kC$-modules*

*(xi)* $\mathrm{Irr}(C)$*: the set of irreducible characters of $C$*

**Proof**

(i) (ii) (iii) (similar for (ix) (x) (xi)): Note that $G$ is a $p'$-group. See [4, Theorem 6.37 of Chapter III].

(iii) (iv) (similar for (vi) (viii)): A block of $G$ is covered by a unique block of $E$, see [4, Corollary 5.6 of Chapter 5]. Hence, when a block $b$ of $G$ is $P$-invariant, $b$ is also a block of $E$, and when a block $b$ of $G$ is not $P$-invariant, $\mathrm{Tr}_1^P(b)$ is a block of $E$. All the blocks of $E$ appear in this way, since a block of $E$ covers some block of $G$, see [4, Lemma 5.3 of Chapter 5]. In the former case, $b$ has a defect group $P$, since the irreducible character of $G$ in $b$ is $P$-invariant and so has $p$ distinct extensions to the irreducible character of $E$ in $b$. In the latter case, $\mathrm{Tr}_1^P(b)$ has a defect 0, see [4, Theorem 5.10 of Chapter 5].

(iv) (v) (similar for (vi) (vii)): Since $kEb \simeq kGb \otimes_k kP$, see [4, Theorem 7.4 of Chapter 5] or [5, Corollary 50.9], $kEb$ has a unique simple module, which is not projective.

(iv) (vi): Since $N_E(P) = CP$, we can consider the Brauer correspondence, see [4, Theorem 2.15 of Chapter 5] (Brauer's first main theorem).

(viii) (ix): The action of $P$ on $\mathrm{Bl}(C)$ is trivial. □

Note that the above correspondences in (v) (ii) and in (vii) (x) are given by the restriction.

**Proposition 3.3.** (Dade) *The correspondence in Lemma 3.2 (i) (xi) is the Glauberman correspondence.*

**Proof**

Let $\chi \in \mathrm{Irr}(G)^P$. Let $b$ be the corresponding block (Lemma 3.2(i) (iv)), and let $\hat{V}$ be the unique simple $kEb$-module, which has a vertex $P$ (Lemma 3.2 (iv) (v)). Let $T$ be a source $kP$-module of $\hat{U}$, see Proposition 3.1. Let $c = \mathrm{Br}_P(b)$ (Lemma 3.2 (iv) (vi)), and let $\hat{U}$ be the unique simple $kCPc$-module, which has a vertex $P$ (Lemma 3.2 (vi) (vii)). Note that $U = \hat{U}\downarrow_C^{CP}$ is the unique simple $kCc$-module (Lemma 3.2 (vii) (x)).

Denoting $\hat{W}$ the Green correspondence of $\hat{V}$, we have

$$\hat{V}\downarrow_{CP}^{E} \simeq \hat{W} \oplus (\text{projective modules}).$$

Note that $\hat{W}$ is in $c$ by the "module version" of the Brauer's second main theorem, see [1, Theorem 3 of Chapter 14], and is a uniserial module whose composition factors are all isomorphic to $\hat{U}$.

If $T \simeq k$, then $\hat{W} \simeq \hat{U}$, and if $T \simeq \Omega_k(k)$, then $\hat{W}$ is an indecomposable $kCPc$-module of length $p-1$.

Note that any indecomposable projective $kCP$-module is a uniserial module of length $p$ whose composition factors are isomorphic, and that $kC$ is semisimple.

Hence, when $T \simeq k$, we have

$$V\downarrow_C^G \simeq \big(\hat{V}\downarrow_G^E\big)\downarrow_C^G \simeq \big(\hat{V}\downarrow_{CP}^E\big)\downarrow_C^{CP} \simeq U \oplus p(\cdots),$$

and when $T \simeq \Omega_k(k)$, we have

$$V\downarrow_C^G \simeq \big(\hat{V}\downarrow_G^E\big)\downarrow_C^G \simeq \big(\hat{V}\downarrow_{CP}^E\big)\downarrow_C^{CP} \simeq (p-1)U \oplus p(\cdots).$$

Hence, we have the proposition. □

Under the above notations, $\hat{V}(P) \simeq \hat{U}$ as modules over $N_E(P) = CP$ by Proposition 2.1, and $\hat{V}(P) \simeq \hat{U}\downarrow_C \simeq U$ as modules over $N_E(P)/P \simeq C$. That is:

**Proposition 3.4.** *We have $\hat{V}(P) \simeq U$, and the correspondence in Lemma 3.2(v) (x) is given by the Brauer construction of the modules with respect to $P$.*

Hence for a $P$-invariant simple $\mathcal{K}G$-module we can get the Glauberman corresponding $\mathcal{K}C$-module by the following procedure:

(i) considering $\mathcal{O}G$-lattice and reduction modulo $p$
(ii) extension to $kE$-module
(iii) Brauer construction of the module with respect to $P$
(iv) $\mathcal{O}$-lift and $\mathcal{K}$-extension

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