Glauberman correspondence as a Brauer construction

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Abstract

We introduce a particular case of Dade's interpretation of the Glauberman correspondence in terms of the modular representation theory of finite groups, and remark that the Glauberman correspondence can be viewed as a module correspondence given by the Brauer construction of the module.

Keywords: finite group; Glauberman correspondence; Brauer construction; Brauer correspondence; Green correspondence

1

For a prime $p$, let $(K, O, k)$ be a $p$-modular system where $O$ is a complete discrete valuation ring having an algebraically closed residue field $k$ of characteristic $p$ and having a quotient field $K$ of characteristic zero which will be assumed to be large enough for any of finite groups we consider in this article. Let $G$ be a finite group.

For the standard facts on the modular representation theory of finite groups, see [1], [4] and [5].

2

In this section, we give a result on the Brauer construction of the module.

Proposition 2.1. Let $P$ be a Sylow $p$-subgroup of $G$, and assume that $P$ has an order $p$. Let $V$ be a simple $kG$-module with a vertex $P$. Then $V(P)$, the Brauer construction of $V$ viewed as a $kN_G(P)$-module, is simple and is isomorphic to the socle of the Green correspondence of $V$.

Proof

By the Green correspondence in a T.I. situation, we have $V^G_{\ast N_G(P)} \simeq W \oplus Y$ where $W$ is an indecomposable module with a vertex $P$ and $Y$ is a projective module, see [1, Theorem 1 of Chapter 10].

Note that $Y(P) = 0$, see [5, Proposition 27.9].

We have $\text{Tr}_1^P(W) = 0$. In fact, since the indecomposable projective $kN_G(P)$-modules are uniserial, see [1, the last part of Chapter 5, or Theorem 1 of Chapter 19], we have $W \subseteq \text{rad } R$ for an indecomposable projective $kN_G(P)$-module $R$. Since $\text{rad } R = (1 - u)R$ for a generator $u$ of $P$, see [1, Lemma 8 of Chapter 5], we have $(1 + u + \cdots + u^{p-1}) \text{rad } R = 0$. Hence we have $(1 + u + \cdots + u^{p-1})W = 0$ and the assertion follows.

Hence we have

\[
V(P) \simeq (V^G_{\ast N_G(P)}(P)) \\
\simeq (W \oplus Y)(P) \\
\simeq W(P) \oplus Y(P) \\
\simeq W(P) \\
\simeq W^P/\text{Tr}_1^P(W) \\
\simeq W^P.
\]
We have
\[ W^P = \text{soc}(W \downarrow_P^{NG(P)}) , \]
since for an indecomposable \( kP \)-module \( X \) we have \( X^P = \text{soc} X \). In fact, we have \( \text{soc} X \subseteq X^P \) and \( 1 = \dim_k \text{soc} X \leq \dim_k X^P \leq \dim_k (kP)^P = 1 \). Note that if \( W \downarrow_P = \oplus_i X_i \), then \( W^P = (\oplus_i X_i)^P = \oplus_i X_i^P \) and \( \text{soc}(W \downarrow_P) = \text{soc}(\oplus_i X_i) = \oplus_i \text{soc}(X_i) \).

Moreover, using [1, Lemma 8 of Chapter 5] recursively, we have
\[ \text{soc}(W \downarrow_P^{NG(P)}) = \text{soc}(W) . \]

Hence, we have the proposition. □

3

In this section, we introduce a particular case of Dade’s interpretation of the Glauberman correspondence (see [3]) in terms of the modular representation theory of finite groups (see Section 13 of [2] for a more general statement). Then we remark that the Glauberman correspondence can be viewed as a module correspondence given by the Brauer construction of the module.

We cite a very particular case of Dade’s Theorem on the endo-permutation modules, which is essential in the proof of Proposition 3.3:

**Proposition 3.1.** (Dade [2] or see [5, Theorem 30.5, Proposition 28.2, Corollary 28.11])

(i) The sources of simple modules of \( p \)-nilpotent groups are endo-permutation modules.

(ii) Endo-permutation \( kP \)-modules for a group \( P \) of order \( p \) are \( k \) and \( \Omega_k(k) \).

Below, we assume that \( G \) has an order not divisible by \( p \). Let \( P \) be a group of order \( p \) acting on \( G \), and \( E \) be a semidirect product of \( G \) and \( P \) with this action. Let \( C = C_G(P) \). We have \( N_E(P) = C_E(P) = CP \).

The following is standard, and below we consider the correspondence suggested in the proof of Lemma 3.2:

**Lemma 3.2.** There is a one-to-one correspondence between the following sets:

(i) \( \text{Irr}(G)^P \): the set of \( P \)-invariant irreducible characters of \( G \)

(ii) \( \text{Simp}(G)^P \): the set of \( P \)-invariant simple \( kG \)-modules

(iii) \( \text{Bl}(G)^P \): the set of \( P \)-invariant \( p \)-blocks of \( G \)

(iv) \( \text{Bl}(E|P) \): the set of \( p \)-blocks of \( E \) with a defect group \( P \)

(v) \( \text{Simp}(E|P) \): the set of simple \( kE \)-modules with a vertex \( P \)

(vi) \( \text{Bl}(C|P)^P \): the set of \( p \)-blocks of \( CP \) with a defect group \( P \)

(vii) \( \text{Simp}(C|P)^P \): the set of simple \( kCP \)-modules with a vertex \( P \)

(viii) \( \text{Bl}(C)^P \): the set of \( P \)-invariant \( p \)-blocks of \( C \)

(ix) \( \text{Bl}(C) \): the set of \( p \)-blocks of \( C \)

(x) \( \text{Simp}(C) \): the set of simple \( kC \)-modules

(xi) \( \text{Irr}(C) \): the set of irreducible characters of \( C \)
Proof
(i) (ii) (iii) (similar for (ix) (x) (xi)): Note that $G$ is a $p'$-group. See [4, Theorem 6.37 of Chapter III].

(iii) (iv) (similar for (vi) (viii)): A block of $G$ is covered by a unique block of $E$, see [4, Corollary 5.6 of Chapter 5]. Hence, when a block $b$ of $G$ is $P$-invariant, $b$ is also a block of $E$, and when a block $b$ of $G$ is not $P$-invariant, $\hat{\text{Tr}}_E(b)$ is a block of $E$. All the blocks of $E$ appear in this way, since a block of $E$ covers some block of $G$, see [4, Lemma 5.3 of Chapter 5]. In the former case, $b$ has a defect group $P$, since the irreducible character of $G$ in $b$ is $P$-invariant and so has $p$ distinct extensions to the irreducible character of $E$ in $b$. In the latter case, $\hat{\text{Tr}}_E(b)$ has a defect 0, see [4, Theorem 5.10 of Chapter 5].

(iv) (v) (similar for (vi) (vii)): Since $kEb \cong kGb \otimes_k kP$, see [4, Theorem 7.4 of Chapter 5] or [5, Corollary 50.9], $kEb$ has a unique simple module, which is not projective.

(iv) (vi): Since $N_E(P) = CP$, we can consider the Brauer correspondence, see [4, Theorem 2.15 of Chapter 5] (Brauer’s first main theorem).

(vii) (ix): The action of $P$ on $\text{Bl}(C)$ is trivial. □

Note that the above correspondences in (v) (ii) and in (vii) (x) are given by the restriction.

**Proposition 3.3.** (Dade) The correspondence in Lemma 3.2 (i) (xi) is the Glauberman correspondence.

**Proof**
Let $\chi \in \text{Irr}(G)^P$. Let $b$ be the corresponding block (Lemma 3.2(i) (iv)), and let $\hat{V}$ be the unique simple $kEb$-module, which has a vertex $P$ (Lemma 3.2 (iv) (v)). Let $T$ be a source $kP$-module of $\hat{U}$, see Proposition 3.1. Let $c = \text{Br}_P(b)$ (Lemma 3.2 (vi) (vi)), and let $\hat{U}$ be the unique simple $kCP$-module, which has a vertex $P$ (Lemma 3.2 (vi) (vii)). Note that $U = \hat{U} \downarrow^E_C$ is the unique simple $kC$-module (Lemma 3.2 (vii) (x)).

Denoting $\hat{W}$ the Green correspondence of $\hat{V}$, we have

$$\hat{V} \downarrow^E_{CP} \cong \hat{W} \oplus (\text{projective modules}).$$

Note that $\hat{W}$ is in $c$ by the “module version” of the Brauer’s second main theorem, see [1, Theorem 3 of Chapter 14], and is a uniserial module whose composition factors are all isomorphic to $\hat{U}$.

If $T \simeq k$, then $\hat{W} \cong \hat{U}$, and if $T \simeq \Omega_k(k)$, then $\hat{W}$ is an indecomposable $kCP$-module of length $p - 1$.

Note that any indecomposable projective $kCP$-module is a uniserial module whose composition factors are isomorphic, and that $kC$ is semisimple.

Hence, when $T \simeq k$, we have

$$V \downarrow_C \cong (\hat{V} \downarrow_{\text{Br}_G}) \downarrow_C \cong (\hat{V} \downarrow^E_{CP}) \downarrow_C \cong U \oplus p(\cdot),$$

and when $T \simeq \Omega_k(k)$, we have

$$V \downarrow_C \cong (\hat{V} \downarrow^E_{CP}) \downarrow_C \cong (\hat{V} \downarrow^E_{CP}) \downarrow_C \cong (p - 1)U \oplus p(\cdot).$$

Hence, we have the proposition. □

Under the above notations, $\hat{V}(P) \simeq \hat{U}$ as modules over $N_E(P) = CP$ by Proposition 2.1, and $\hat{V}(P) \simeq U \downarrow_C \simeq U$ as modules over $N_E(P)/P \cong C$. That is:

**Proposition 3.4.** We have $\hat{V}(P) \cong U$, and the correspondence in Lemma 3.2(v) (x) is given by the Brauer construction of the modules with respect to $P$.

Hence for a $P$-invariant simple $K$-module we can get the Glauberman corresponding $K$-module by the following procedure:

(i) considering $OG$-lattice and reduction modulo $p$
(ii) extension to $kE$-module
(iii) Brauer construction of the module with respect to $P$
(iv) $O$-lift and $K$-extension
References


